

A New 2d/4d Duality via Integrability

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Abstract

We prove a duality, recently conjectured in arXiv:1103.5726, which relates the F-terms of supersymmetric gauge theories defined in two and four dimensions respectively. The proof proceeds by a saddle point analysis of the four-dimensional partition function in the Nekrasov-Shatashvili limit. At special quantized values of the Coulomb branch moduli, the saddle point condition becomes the Bethe Ansatz Equation of the $SL(2)$ Heisenberg spin chain which coincides with the F-term equation of the dual two-dimensional theory. The on-shell values of the superpotential in the two theories are shown to coincide in corresponding vacua. We also identify two-dimensional duals for a large set of quiver gauge theories in four dimensions and generalize our proof to these cases.

1 Introduction

Two dimensional theories have long been studied as toy models for aspect of four-dimensional gauge dynamics such as asymptotic freedom, instanton effects, the generation of a mass gap and large- N limits. Recently a duality between two- and four-dimensional theories was conjectured [1] which makes this analogy precise for some protected quantities in the supersymmetric setting. The proposed duality relates four-dimensional $\mathcal{N} = 2$ gauge theories in a particular Ω background to $\mathcal{N} = (2, 2)$ gauged linear sigma models in two dimensions. The new duality extends an earlier proposal [2–4] which related the BPS spectrum of $\mathcal{N} = (2, 2)$ QED with charged matter to that of $SU(N)$ Seiberg-Witten theory with massive flavors at the Higgs branch root. In particular it can be regarded as an extension of the earlier proposal away from the Higgs branch root which holds at a generic point on the Coulomb branch of the four-dimensional theory. The proposal also makes contact with another, quite different type of 2d/4d duality: the AGT conjecture [5] which relates the instanton partition functions of four-dimensional $\mathcal{N} = 2$ superconformal theories to conformal blocks of Liouville theories on Riemann surfaces. In this letter, we will present a proof of the conjecture of [1] and also extend the duality to a larger class of $\mathcal{N} = 2$ quiver gauge theories in four dimensions.

Let us begin by recalling the two specific theories of the aforementioned duality, which we shall refer to as Theory I and II.

Theory I: Four Dimensional $\mathcal{N} = 2$ SQCD with gauge group $SU(L)$, with L fundamental hypermultiplets of masses $\vec{m}_F = (m_1, \dots, m_L)$ and L anti-fundamental hypermultiplets of masses $\vec{m}_{AF} = (\tilde{m}_1, \dots, \tilde{m}_L)$. The marginal coupling constant is $\tau = 4\pi i/g^2 + \vartheta/2\pi$.

Theory I is now also subjected to a particular Nekrasov deformation on one-plane with the deformation parameters $(\epsilon_1, \epsilon_2) = (\epsilon, 0)$, which preserves $\mathcal{N} = (2, 2)$ supersymmetry in a two-dimensional subspace of four-dimensional space-time [6]. This Nekrasov deformation, or Ω -background, turns out to lift the Coulomb branch moduli space of the given theory, leaving isolated vacua at the points,

$$\vec{a} = \vec{m}_F - \vec{n}\epsilon, \tag{1.1}$$

where \vec{a} are the usual special Kähler coordinates on the Coulomb branch and $\vec{n} = (n_1, n_2, \dots, n_L) \in \mathbb{Z}^L$. In the presence of the deformation, the partition function of Nekrasov provides a twisted superpotential $\mathcal{W}^{(I)}$ that describes the low-energy dynamics of Theory I and whose critical points are given by (1.1).

The other system of interest is,

Theory II: Two dimensional $\mathcal{N} = (2, 2)$ supersymmetric Yang-Mills with gauge group $U(N)$, with L fundamental chiral multiplets with twisted masses $\vec{M}_F = (M_1, \dots, M_L)$ and L anti-fundamental chiral multiplets with twisted masses $\vec{M}_{AF} = (\tilde{M}_1, \dots, \tilde{M}_L)$ as well as a single adjoint chiral multiplet with twisted mass ϵ . The FI parameter r and 2d vacuum angle θ also combine to give a holomorphic coupling constant $\hat{\tau} = ir + \theta/2\pi$.

As explained in [1], Theory II arises as the worldvolume theory of surface operators/vortex strings which probe the Higgs branch of Theory I. The low-energy dynamics of Theory II is also characterized by a twisted superpotential $\mathcal{W}^{(II)}$ whose vacuum conditions takes the following form

$$\prod_{l=1}^L \frac{\lambda_j - M_l}{\lambda_j - \tilde{M}_l} = -q \prod_{k=1}^N \frac{\lambda_j - \lambda_k - \epsilon}{\lambda_j - \lambda_k + \epsilon}, \quad q = (-1)^{N+1} e^{2\pi i \hat{\tau}}. \quad (1.2)$$

Here $\{\lambda_j\}$ represent vacuum expectation values of the scalar field in the vector multiplet, while the condition (1.2) coincides with the Bethe Ansatz Equations (BAEs) of the $SL(2)$ Heisenberg spin chain, $\{\lambda_i\}$ being associated with magnon rapidities or “Bethe roots”. The vacuum equation allows non-degenerate vacua, parameterized again by a set of integers \hat{n}_l with $N = \sum_{l=1}^L \hat{n}_l$, whose weak-coupling expressions become

$$\lambda_{(ls)} = M_l - (s-1)\epsilon + \mathcal{O}(q), \quad s = 1, \dots, \hat{n}_l. \quad (1.3)$$

According to [7], massive theories preserving two-dimensional $\mathcal{N} = (2, 2)$ supersymmetry can be classified by their critical values of (twisted) superpotentials^{#1}. In [1], it has been checked, up to first few orders of instanton expansion in $q = e^{2\pi i \tau}$ that there is an one-to-one correspondence between the supersymmetric vacua of two theories. Moreover it has been further conjectured that the on-shell values of their twisted superpotentials

^{#1}A massive theory is defined to have a mass gap with non-degenerate vacua. Authors have also discussed a refined classification of two-dimensional theories by their degeneracies of BPS spectra.

coincide:

$$\mathcal{W}^{(\text{I})}(a_l = m_l - n_l \epsilon) - \mathcal{W}^{(\text{I})}(a_l = m_l - \epsilon) \equiv \mathcal{W}^{(\text{II})}(\{\hat{n}_l\}) , \quad (1.4)$$

provided the parameters in both theories are identified as follows ^{#2}:

$$\hat{\tau} = \tau + \frac{1}{2}(N+1) , \quad \vec{M}_F = \vec{m}_F - \frac{3}{2}\epsilon , \quad \vec{M}_{AF} = \vec{m}_{AF} + \frac{1}{2}\epsilon \quad (1.5)$$

with $\hat{n}_l = n_l - 1$. In other words, protected holomorphic structures of Theory I and Theory II are isomorphic. In particular, two theories have the same chiral ring structure. The explicit identifications between the chiral rings of the two theories will be discussed further below.

Four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories in the Ω -background with $(\epsilon_1, \epsilon_2) = (\epsilon, 0)$ have been studied by Nekrasov and Shatashvili [6] in relation to the quantum integrable systems. In particular, the generators of the twisted chiral ring are mapped to quantum Hamiltonians. It is known that the Seiberg-Witten curve of Theory I is nothing but the spectral curve of the classical $SL(2, \mathbb{R})$ spin chain [10, 11]. As above, the vacuum equations (1.2) of Theory II can be identified as the BAEs of the same spin chain, where the parameter ϵ plays a role as the Planck constant \hbar . The duality therefore supports the idea of Nekrasov and Shatashvili and may shed new light on the quantisation of integrable systems.

In order to prove this duality (1.4), we rely on the saddle point analysis of the Nekrasov partition function of Theory I in the $\epsilon_2 \rightarrow 0$ limit, developed recently in [12–14]. More precisely, we will see how the Bethe Ansatz Equation (BAE) of $SL(2, \mathbb{R})$ spin chain can arise from the saddle point equations of the instanton partition function. As a consequence we can also show that the on-shell Nekrasov partition function, $\mathcal{W}^{(\text{I})}$ agrees with the on-shell Yang-Yang potential [15] of $SL(2, \mathbb{R})$ spin chain, $\mathcal{W}^{(\text{II})}$. Applying the same analysis, we can prove the duality for a large class of linear quiver gauge theories.

^{#2}The second term in (1.4) is a vacuum independent subtraction which ensures that the superpotential vanishes at the Higgs branch root.

2 BAE from Nekrasov Instanton Partition Function

There are several ways to present Nekrasov's extraordinary result for the instanton partition function of an $\mathcal{N} = 2$ gauge theory. Our starting point will be the gamma function representation for the instanton partition function in the $\mathcal{N} = 2$ gauge theory with $2L$ fundamental hypermultiplets [9]. The expression depends on a sum over L Young Tableaux $\vec{Y} = (Y_1, \dots, Y_L)$. The number of boxes in i^{th} row of the tableau Y_l ($l = 1, 2, \dots, L$) is denoted k_{li} and $|\vec{Y}|$ is the total number of boxes in all the L tableaux. In the following, $q = e^{2\pi i \tau}$ is the coupling and we have defined (following [13])

$$x_{li} = a_l + (i-1)\epsilon_1 + \epsilon_2 k_{li}, \quad x_{li}^{(0)} = a_l + (i-1)\epsilon_1, \quad (2.1)$$

where i, j , etc., are indices that range from 1 to ∞ . The partition function involves a sum over the L tableaux,

$$\mathcal{Z}_{\text{inst}} = \sum_{\vec{Y}} q^{|\vec{Y}|} \mathcal{Z}_{\text{vec}}(\vec{Y}) \prod_{n=1}^{2L} \mathcal{Z}_{\text{hyp}}(\vec{Y}, \mu_n), \quad (2.2)$$

where the contribution from the vector multiplet can be written

$$\mathcal{Z}_{\text{vec}}(\vec{Y}) = \prod_{(li) \neq (nj)} \frac{\Gamma(\epsilon_2^{-1}(x_{li} - x_{nj} - \epsilon_1))}{\Gamma(\epsilon_2^{-1}(x_{li} - x_{nj}))} \cdot \frac{\Gamma(\epsilon_2^{-1}(x_{li}^{(0)} - x_{nj}^{(0)}))}{\Gamma(\epsilon_2^{-1}(x_{li}^{(0)} - x_{nj}^{(0)} - \epsilon_1))} \quad (2.3)$$

and the contribution from a single fundamental hypermultiplet of mass μ is

$$\mathcal{Z}_{\text{hyp}}(\vec{Y}, \mu) = \prod_{li} \frac{\Gamma(\epsilon_2^{-1}(x_{li} + \mu))}{\Gamma(\epsilon_2^{-1}(x_{li}^{(0)} + \mu))}. \quad (2.4)$$

In order to agree with the conventions of [1], we take our $2L$ hypermultiplets to have masses $\{-m_l + \epsilon_1, -\tilde{m}_l\}$.

Now we consider the Nekrasov-Shatashvili limit $\epsilon_2 \rightarrow 0$ with $\epsilon \equiv \epsilon_1$ fixed. In this limit, we can approximate the gamma functions using Stirling's approximation, to find the leading order behaviour

$$\mathcal{Z}_{\text{vec}}(\vec{Y}) = \exp \left[\frac{1}{2\epsilon_2} \sum_{(li) \neq (nj)} \left(f(x_{li} - x_{nj} - \epsilon) - f(x_{li} - x_{nj} + \epsilon) \right. \right. \\ \left. \left. - f(x_{li}^{(0)} - x_{nj}^{(0)} - \epsilon) + f(x_{li}^{(0)} - x_{nj}^{(0)} + \epsilon) \right) \right] \quad (2.5)$$

and

$$\mathcal{Z}_{\text{hyp}}(\vec{Y}, \mu) = \exp \left[\frac{1}{\epsilon_2} \sum_{li,n} \left(f(x_{li} + \mu) - f(x_{li}^{(0)} + \mu) \right) \right] , \quad (2.6)$$

where $f(x) = x(\log x - 1)$. The coupling constant piece can then be written as

$$q^{|\vec{Y}|} = \exp \left[\frac{\log q}{\epsilon_2} \sum_{li} (x_{li} - x_{li}^{(0)}) \right] . \quad (2.7)$$

In the NS limit, $\epsilon_2 k_{li}$ becomes continuous and so the sum over Young Tableaux can be traded for an integral over the infinite set of variables $\{x_{li}\}$ and we can write

$$\mathcal{Z}_{\text{inst}} = \int \prod_{li} dx_{li} \exp \left[\frac{1}{\epsilon_2} \mathcal{H}_{\text{inst}}(x_{li}) \right] , \quad (2.8)$$

where the instanton action functional takes the difference form

$$\mathcal{H}_{\text{inst}}(x_{li}) = \mathcal{Y}(x_{li}) - \mathcal{Y}(x_{li}^{(0)}) , \quad (2.9)$$

where

$$\begin{aligned} \mathcal{Y}(x_{li}) = & \log q \sum_{li} x_{li} + \sum_{li,n} (f(x_{li} - \tilde{m}_n) + f(x_{li} - m_n + \epsilon)) \\ & + \frac{1}{2} \sum_{(li) \neq (nj)} (f(x_{li} - x_{nj} - \epsilon) - f(x_{li} - x_{nj} + \epsilon)) . \end{aligned} \quad (2.10)$$

In order to make contact with [6, 13], we can write the instanton action functional in integral form by introducing the instanton “density” $\rho(x)$ which is constant along the series of intervals

$$\mathcal{J} = \bigcup_{li} [x_{li}^{(0)}, x_{li}] . \quad (2.11)$$

More precisely, these are contours in the complex plane with end points x_{li} and $x_{li}^{(0)}$. Then using the identity

$$\sum_{i=1}^{\infty} (f(y - x_{li}^{(0)} - \epsilon) - f(y - x_{li}^{(0)} + \epsilon)) = f(y - a_l - \epsilon) + f(y - a_l) , \quad (2.12)$$

one can show

$$\mathcal{H}_{\text{inst}}[\rho] = -\frac{1}{2} \int dx dy \rho(x) \mathfrak{G}(x - y) \rho(y) + \int dx \rho(x) \log (q \Re(x)) . \quad (2.13)$$

Here, the integration kernel is given by

$$\mathfrak{G}(x) = \frac{d}{dx} \log \left(\frac{x - \epsilon}{x + \epsilon} \right) \quad (2.14)$$

and

$$\mathfrak{R}(x) = \frac{A(x)D(x + \epsilon)}{P(x)P(x + \epsilon)} , \quad (2.15)$$

with

$$A(x) = \prod_{l=1}^L (x - \tilde{m}_l) , \quad D(x) = \prod_{l=1}^L (x - m_l) , \quad P(x) = \prod_{l=1}^L (x - a_l) . \quad (2.16)$$

In the Nekrasov-Shatashvili limit $\epsilon_2 \rightarrow 0$, the functional integral (2.8) is dominated by a saddle point configuration; variation of the instanton density $\rho(x)$ can be effectively achieved by small variation of end points x_{li} of \mathcal{J} where $\rho(x)$ should remain constant. The saddle point equation then becomes

$$\frac{\delta \mathcal{H}_{\text{inst}}[\rho]}{\delta x_{li}} = - \int_{\mathcal{J}} dy \mathfrak{G}(x_{li} - y) \rho(y) + \log(q \mathfrak{R}(x_{li})) = 0 . \quad (2.17)$$

Since $\mathfrak{G}(x)$ is a total derivative, we can easily rewrite the above equation into a following form

$$\frac{\mathfrak{Q}(x_{li} + \epsilon) \mathfrak{Q}^{(0)}(x_{li} - \epsilon)}{\mathfrak{Q}(x_{li} - \epsilon) \mathfrak{Q}^{(0)}(x_{li} + \epsilon)} = -q \mathfrak{R}(x_{li}) \quad (2.18)$$

where

$$\mathfrak{Q}(x) = \prod_{l=1}^L \prod_{i=1}^{\infty} (x - x_{li}) , \quad \mathfrak{Q}^{(0)}(x) = \prod_{l=1}^L \prod_{i=1}^{\infty} (x - x_{li}^{(0)}) . \quad (2.19)$$

Using the explicit expression for $x_{li}^{(0)}$ in (2.1), one can further simplify the saddle point equation as follows

$$\frac{\mathfrak{Q}(x_{li} + \epsilon)}{\mathfrak{Q}(x_{li} - \epsilon)} = -q A(x_{li}) D(x_{li} + \epsilon) . \quad (2.20)$$

The above equations (2.20) are an infinite set of equations for the end-points x_{li} of intervals \mathcal{J} . Notice that these equation do not depend on $x_{li}^{(0)}$; however, they must be solved subject to the condition that the solution has the expansion

$$x_{li}^{(0)} = a_l + (i - 1)\epsilon_1 , \quad x_{li} = x_{li}^{(0)} + \sum_{k=i}^{\infty} q^k x_{li}^{(k)} \quad i = 1, \dots, \infty , \quad (2.21)$$

in order that the instanton partition function has a consistent expansion in q . It is important that order of q correlates with the index i , so that at any given order in the instanton expansion $\mathcal{O}(q^k)$, we can effectively truncate the infinite system of equations by taking $x_{li} = x_{li}^{(0)}$, for $i > k$. It is rather remarkable that the equations (2.20) related to a quantization of the Seiberg-Witten curve that is recovered in the limit $\epsilon \rightarrow 0$ [13,14], which we will discuss later.

The important result we now want to verify is that the infinite set of saddle-point equations has a natural truncation to a finite system if one imposes quantization conditions on the VEVs a_l

$$a_l = m_l - n_l \epsilon, \quad n_l \in \mathbb{Z} > 0. \quad (2.22)$$

More precisely, one can show from (2.22) and (2.20) that most of intervals in \mathcal{J} become degenerate

$$x_{li} = x_{li}^{(0)} = a_l + (i - 1)\epsilon, \text{ for } i \geq n_l, \quad (2.23)$$

which leads to collapsing of the infinite set of saddle point equations onto a finite set of equations. We will present a formal proof of this statement because of its central role in our analysis.

Proof: Following [13], we define

$$w(x) = \frac{\mathfrak{Q}(x - \epsilon)}{\mathfrak{Q}(x)}. \quad (2.24)$$

One can then rewrite the saddle point equation (2.20) as

$$1 + qA(x_{li})D(x_{li} + \epsilon)w(x_{li})w(x_{li} + \epsilon) = 0. \quad (2.25)$$

For later convenience, let us consider a function $T(x)$

$$T(x) = \frac{h+2}{w(x+\epsilon)} \left[1 - \frac{h}{h+2} A(x)D(x+\epsilon)w(x)w(x+\epsilon) \right], \quad (2.26)$$

where $q = -\frac{h}{h+2}$. Using (2.25), we can see that apparent poles in $T(x)$ coming from the zeros of $w(x+\epsilon)$ are cancelled by corresponding zeros in the numerator. It implies that $T(x)$ is analytic in the complex plane. From the asymptotic behavior of $w \sim x^{-L}$

at large x [13], one can conclude that $T(x)$ should be a polynomial of degree L . In the limit $\epsilon \rightarrow 0$, (2.26) reduces to a defining equation of the Seiberg-Witten curve for $\mathcal{N} = 2$ $SU(L)$ SQCD with $N_F = 2L$ fundamental flavours [10, 11]

$$t^2 - T(x)t - h(h+2)A(x)D(x) = 0, \quad t = \frac{h+2}{w(x)}. \quad (2.27)$$

In particular, the coefficients in the polynomial function $T(x)$ correspond to the Coulomb branch moduli. It strongly suggests that, with finite ϵ , (2.26) can now be interpreted as a quantization of the Seiberg-Witten curve [13].

It follows from the above that

$$\mathcal{A}(x + \epsilon) - \mathcal{B}(x)\mathcal{A}(x) = -q\mathfrak{R}(x)\mathcal{A}(x - \epsilon), \quad (2.28)$$

where

$$\mathcal{A}(x) = \frac{\mathfrak{Q}(x)}{\mathfrak{Q}^{(0)}(x)}, \quad \mathcal{B}(x) = \frac{1}{(h+2)} \frac{T(x)}{P(x + \epsilon)} \quad (2.29)$$

Notice that $\mathcal{A}(x)$ has poles at $x_{li}^{(0)}$. Now generically both sides of (2.28), have poles at $a_l + (i-2)\epsilon$, $i = 1, 2, \dots, \infty$. But if the quantization condition (2.22) is imposed then the pole on the right-hand side at $a_l + (n_l - 1)\epsilon$ is missing because then $\mathfrak{R}(x)$ has a zero there. Consequently, on the left-hand side, $\mathcal{A}(x)$ cannot have a pole at $a_l + (n_l - 1)\epsilon$. But then the right-hand side does not have a pole at $a_l + n_l\epsilon$ implying on the left-hand side $\mathcal{A}(x)$ cannot have a pole at $a_l + n_l\epsilon$. The argument continues inductively for $i \geq n_l$ and the conclusion is that $\mathcal{A}(x)$ only has a finite set of poles at $a_l + (i-1)\epsilon$, for $i = 1, 2, \dots, n_l - 1$. This implies that

$$x_{li} = x_{li}^{(0)} = a_l + (i-1)\epsilon, \text{ for } i \geq n_l, \quad (2.30)$$

so only the first $n_l - 1$ rows of the Young tableau Y_l are occupied. This completes the proof. ■

As a consequence of the above truncation, the quantised Seiberg-Witten curve indeed can be identified as the Baxter equation of our interest. The details of it are in order. Defining a finite polynomial

$$\hat{\mathfrak{Q}}(x) = \prod_{l=1}^L \prod_{i=1}^{n_l-1} (x - x_{li}), \quad (2.31)$$

one can show that

$$\begin{aligned} w(x) &= \frac{\hat{\mathfrak{Q}}(x - \epsilon)}{\hat{\mathfrak{Q}}(x)} \prod_{l=1}^L \frac{1}{x - a_l - (n_l - 1)\epsilon} \\ &= \frac{\hat{\mathfrak{Q}}(x - \epsilon)}{\hat{\mathfrak{Q}}(x)} \cdot \frac{1}{D(x + \epsilon)} , \end{aligned} \quad (2.32)$$

where we used for the last equality the quantisation condition (2.22). The quantised Seiberg-Witten curve (2.26) can then be simplified as follows

$$T(x)\hat{\mathfrak{Q}}(x) = (h + 2)D(x + 2\epsilon)\hat{\mathfrak{Q}}(x + \epsilon) - hA(x)\hat{\mathfrak{Q}}(x - \epsilon) , \quad (2.33)$$

while the saddle point equations (2.25) become

$$\frac{D(x_{li} + 2\epsilon)}{A(x_{li})} = -q \frac{\hat{\mathfrak{Q}}(x_{li} - \epsilon)}{\hat{\mathfrak{Q}}(x_{li} + \epsilon)} . \quad (2.34)$$

In order to make the identification of two equations (2.33, 2.34) with the Baxter equation and BAE of the $SL(2, \mathbb{R})$, let us apply the identification of the mass parameters given in (1.5) and set $\lambda = x + \frac{1}{2}\epsilon$. It leads to

$$\hat{\mathfrak{Q}}(x) = Q(\lambda) = \prod_{l=1}^L \prod_{i=1}^{n_l-1} (\lambda - \lambda_{li}) , \quad \lambda_{li} = x_{li} + \frac{1}{2}\epsilon , \quad (2.35)$$

and

$$A(x) = a(\lambda) = \prod_{l=1}^L (\lambda - \tilde{M}_l) , \quad D(x + 2\epsilon) = d(\lambda) = \prod_{l=1}^L (\lambda - M_l) . \quad (2.36)$$

One can finally show that (2.33) can be rewritten as a standard form of the Baxter equation for the spin chain

$$t(\lambda)Q(\lambda) = (h + 2)d(\lambda)Q(\lambda + \epsilon) - ha(\lambda)Q(\lambda - \epsilon) , \quad (2.37)$$

where $t(\lambda) = T(x)$ can be understood as the eigenvalue of the spin chain transfer matrix. One can also see that (2.26) are precisely the BAE of $SL(2, \mathbb{R})$ spin chain

$$\frac{d(\lambda_{li})}{a(\lambda_{li})} = -q \frac{Q(\lambda_{li} - \epsilon)}{Q(\lambda_{li} + \epsilon)} . \quad (2.38)$$

It is noteworthy here that the finite instanton string $x_{li}^{(0)} = a_l + (i-1)\epsilon$ ($i = 1, \dots, n_l - 1$) can be identified with the classical Bethe string solution,

$$\lambda_{(ls)}^{(0)} = M_l - (s-1)\epsilon, \quad s = 1, 2, \dots, \hat{n}_l, \quad (2.39)$$

with $\hat{n}_l = n_l - 1$ and $s = n_l - i$.

By explicit evaluation of the instanton action with the quantization condition and subsequent truncation, we can go one step further to show how the Yang-Yang functional $Y(\lambda_j)$ of the spin chain (2.38), twisted superpotential of the two-dimensional theory, can arise from the above analysis. Denoting $N = \sum_{l=1}^L n_l - 1$, it follows from (2.9) that the instanton action in the truncated theory takes the form

$$\mathcal{W}_{\text{inst}}^{(\text{I})}(m_l - n_l\epsilon) = \hat{\mathcal{Y}}(x_{li}) - \hat{\mathcal{Y}}(x_{li}^{(0)}), \quad (2.40)$$

where the function $\hat{\mathcal{Y}}(x)$ is a truncated version of $\mathcal{Y}(x)$

$$\begin{aligned} \hat{\mathcal{Y}}(x_{li}) = & \log q \sum_{(li)=1}^N x_{li} + \sum_{(li)=1}^N \sum_{n=1}^L \left(f(x_{li} - \tilde{m}_n) - f(x_{li} - m_n + 2\epsilon) \right) \\ & + \frac{1}{2} \sum_{(li) \neq (mj)=1}^N \left(f(x_{li} - x_{mj} - \epsilon) - f(x_{li} - x_{mj} + \epsilon) \right). \end{aligned} \quad (2.41)$$

If we make the parameter identification (1.5) and change of variable $\lambda = x + \frac{1}{2}\epsilon$ as before, we can show that

$$\mathcal{W}_{\text{inst}}^{(\text{I})}(m_l - n_l\epsilon) = \mathcal{W}^{(\text{II})}(\lambda_{ls}) - \mathcal{W}^{(\text{II})}(\lambda_{ls}^{(0)}), \quad (2.42)$$

where we have identified

$$\begin{aligned} \mathcal{W}^{(\text{II})}(\lambda_{ls}) \equiv Y(\lambda_{ls}) = & \log q \sum_{(ls)=1}^N \lambda_{ls} + \sum_{(ls)=1}^N \sum_{n=1}^L \left(f(\lambda_{ls} - \tilde{M}_n) - f(\lambda_{ls} - M_n) \right) \\ & + \frac{1}{2} \sum_{(ls) \neq (mp)=1}^N \left(f(\lambda_{ls} - \lambda_{mp} - \epsilon) - f(\lambda_{ls} - \lambda_{mp} + \epsilon) \right) \end{aligned} \quad (2.43)$$

as the Yang-Yang functional for the spin chain [15]. Note that the equations-of-motion of the functional $Y(\lambda_j)$ are the BAE (2.38). Since the instanton contribution to $\mathcal{W}_{\text{inst}}^{(\text{I})}$ at the root of baryonic Higgs branch identically vanishes

$$\mathcal{W}_{\text{inst}}^{(\text{I})}(m_l - \epsilon) = 0, \quad (2.44)$$

the complete matching of the two theories (1.4)

$$\mathcal{W}^{(\text{I})}(m_l - n_l \epsilon) - \mathcal{W}^{(\text{I})}(m_l - \epsilon) = \mathcal{W}^{(\text{II})}(\lambda_{ls}) \equiv Y(\lambda_{ls}) \quad (2.45)$$

requires perturbative contributions to satisfy a following relation

$$\mathcal{W}_{\text{pert}}^{(\text{I})}(m_l - n_l \epsilon) - \mathcal{W}_{\text{pert}}^{(\text{I})}(m_l - \epsilon) = \mathcal{W}^{(\text{II})}(\lambda_{ls}^{(0)}) \equiv Y(\lambda_{ls}^{(0)}) . \quad (2.46)$$

It is rather trivial to see the matching of the classical parts

$$\mathcal{W}_{\text{cl}}^{(\text{I})}(m_l - n_l \epsilon) - \mathcal{W}_{\text{cl}}^{(\text{I})}(m_l - \epsilon) = \log q \sum_{(ls)=1}^N \lambda_{ls}^{(0)} , \quad (2.47)$$

where

$$\mathcal{W}_{\text{cl}}^{(\text{I})}(a_l) = -\frac{\log q}{2\epsilon} \sum_{l=1}^L a_l^2 . \quad (2.48)$$

The one-loop contribution is given by

$$\mathcal{W}_{1\text{-loop}}^{(\text{I})}(a_l) = \sum_{l,n} [\omega_\epsilon(a_l - \tilde{m}_n - \epsilon) + \omega_\epsilon(a_l - m_n) - \omega_\epsilon(a_l - a_n)] , \quad (2.49)$$

where $\omega_\epsilon(x)$ satisfies $\frac{d\omega_\epsilon(x)}{dx} = -\log \Gamma(1 + x/\epsilon)$. It needs much elaboration, discussed in details in [1], to show that

$$\mathcal{W}_{1\text{-loop}}^{(\text{I})}(m_l - n_l \epsilon) - \mathcal{W}_{1\text{-loop}}^{(\text{I})}(m_l - \epsilon) = \mathcal{W}^{(\text{II})}(\lambda_j^{(0)}) - \log q \sum_{(ls)} \lambda_{ls}^{(0)} , \quad (2.50)$$

which completes the proof of the conjectured duality in [1] between in Theories I and II (1.4).

Let us finish this section by commenting on the VEVs of the chiral operators $\hat{\mathcal{O}}_k = \text{Tr} \varphi^k$. It was proposed in [1] that these are related in a simple way to the conserved charges of the associated spin chain which correspond to the coefficients of the polynomial $t(\lambda)$ appearing in the Baxter eqn above. This is a natural generalisation of the usual relation between the corresponding VEVs and the coefficients in the polynomial $T(x)$ appearing in the Seiberg-Witten curve (2.27) of the undeformed $\epsilon = 0$ case. The proposal of [1] can be explicitly recast as,

$$\begin{aligned} \langle \text{Tr} \varphi^k \rangle_{\text{DHL}} &= \int_{\mathcal{C}} \frac{d\lambda}{2\pi i} \lambda^k \frac{d}{d\lambda} \log \left(\frac{Q(\lambda + \epsilon)}{Q(\lambda)} \frac{Q_0(\lambda)}{Q_0(\lambda + \epsilon)} \right) \\ &\quad + \int_{\mathcal{C}} \frac{d\lambda}{2\pi i} \lambda^k \frac{d}{d\lambda} \log \left(1 + q \frac{a(\lambda)Q(\lambda - \epsilon)}{d(\lambda)Q(\lambda + \epsilon)} \right) . \end{aligned} \quad (2.51)$$

On the other hand we can calculate the expectation values directly using the instanton calculus instead. Indeed the Nekrasov partition function with operators $\hat{\mathcal{O}}_k$ inserted can be evaluated readily using the saddle point approach described above [13]. In our notation this yields: ^{#3}

$$\langle \text{Tr} \varphi^k \rangle_{\text{SC}} = \int_{\mathcal{C}} \frac{d\lambda}{2\pi i} \lambda^k \frac{d}{d\lambda} \log \left(\frac{Q(\lambda + \epsilon)}{Q(\lambda)} \frac{Q_0(\lambda)}{Q_0(\lambda + \epsilon)} \right). \quad (2.52)$$

Which reproduces the first term of (2.51) but not the second. Here the contour \mathcal{C} encloses the entire complex plane, hence all the zeros in $Q(\lambda)$, $Q(\lambda + \epsilon)$, $Q_0(\lambda)$ and $Q_0(\lambda + \epsilon)$, $Q_0(\lambda)$ is defined as $Q(\lambda)$, with $\lambda_i \rightarrow \lambda_i^{(0)}$.

For the case of $\mathcal{N} = 2$ SQCD with gauge group $SU(L)$ and $N_F < L$ fundamental flavours the two corresponding definitions were shown to be equivalent in [13] (see in particular Eqn (46) in this reference). However, the equivalence does not hold for $N_F \geq L$ and in particular does not hold in the present case $N_F = 2L$. This reflects a well known ambiguity in parametrising the Coulomb branch first uncovered in [17]. Even in the undeformed case $\epsilon = 0$, it is known that the VEVs extracted from the Seiberg-Witten curve are not equal to those obtained from direct semiclassical calculations but are related to the latter by holomorphic operator mixings which are allowed by the symmetries of the theory. In the present case, the VEVs conjectured in [1] are related by similar holomorphic mixings to those of the direct calculation. The explicit form of these mixings can be deduced from the equations given above but we will not consider these further here.

3 Generalisation to Linear Quiver Theories

We can now apply these ideas to the quiver gauge theories and derive the equations which can be interpreted as the BAE of an associated spin system. Brane constructions of dual four- and two-dimensional theories are shown in Figures (3.1) and (3.2) respectively. As Theory I, we will consider the A_p linear quiver theory in four dimensions with gauge group $SU(L)^p$ and bi-fundamental hypermultiplets between the nodes of mass μ_I , $I =$

^{#3}Here we have removed the perturbative pieces, and taken into account the mapping between the parameters in Theory I and II as given in (1.5).

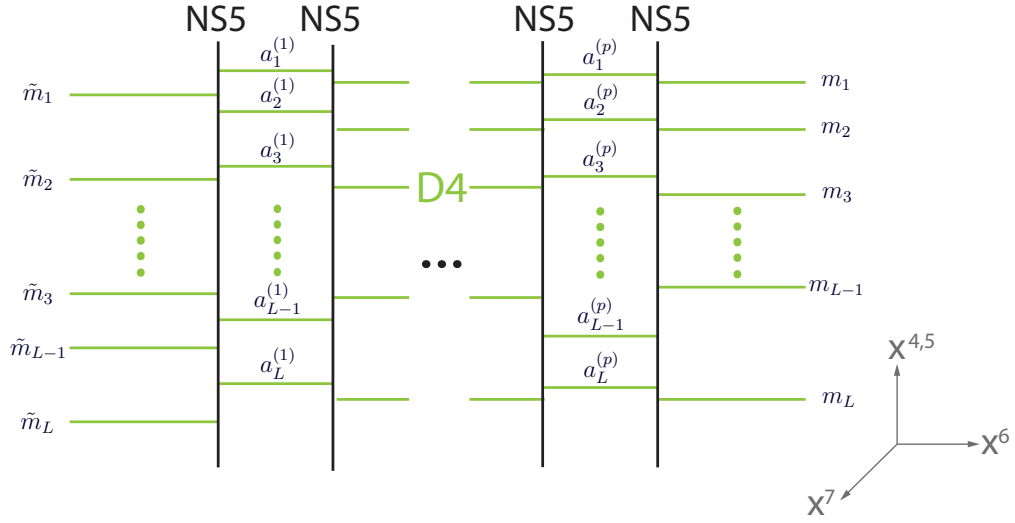


Figure 3.1: The IIA-brane construction for Theory I in the linear quiver case.

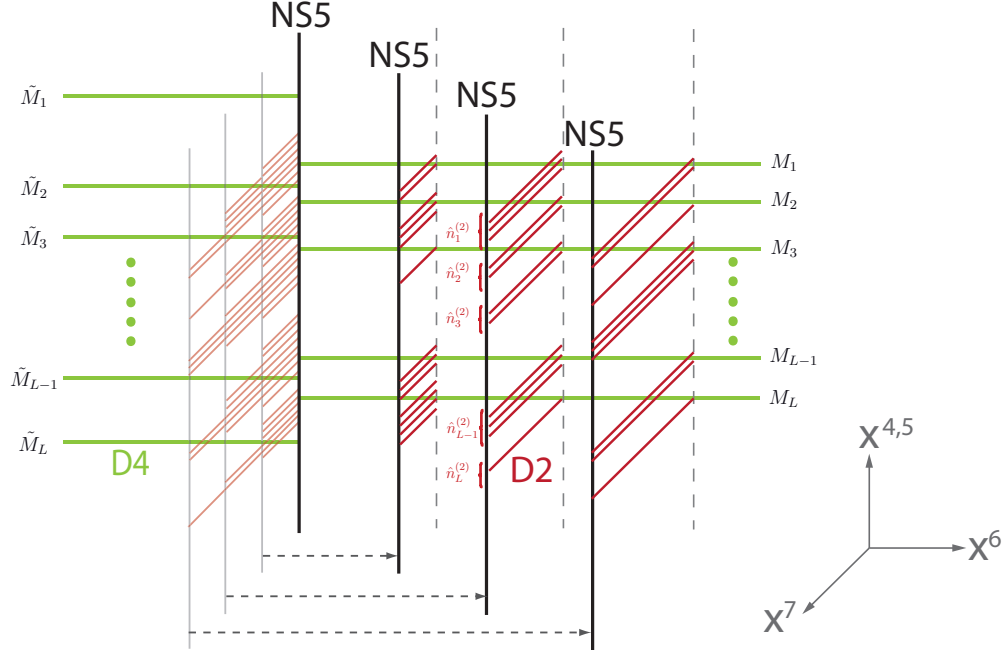


Figure 3.2: The IIA-brane construction for Theory II in the linear quiver case.

$1, \dots, p-1$ and the first and last node have $L(\text{anti-})$ fundamental hypermultiplets of mass $-\tilde{m}_l$ and $-m_l + \epsilon$, respectively. The contribution from a bi-fundamental hypermultiplet of mass μ_I charged under the I^{th} and $I+1^{\text{th}}$ $SU(L)$ factors of the gauge group to the instanton partition function is

$$\begin{aligned} \mathcal{Z}_{\text{bi-fund}}(\vec{Y}) &= \prod_{li, nj} \frac{\Gamma(\epsilon_2^{-1}(x_{li}^{(I)} - x_{nj}^{(I+1)} + \mu_I))}{\Gamma(\epsilon_2^{-1}(x_{li}^{(0,I)} - x_{nj}^{(0,I+1)} + \mu_I))} \cdot \frac{\Gamma(\epsilon_2^{-1}(x_{li}^{(I+1)} - x_{nj}^{(I)} + \epsilon_1 + \epsilon_2 - \mu_I))}{\Gamma(\epsilon_2^{-1}(x_{li}^{(0,I+1)} - x_{nj}^{(0,I)} + \epsilon_1 + \epsilon_2 - \mu_I))} . \end{aligned} \quad (3.1)$$

There is a subtlety here, explained in [5], that the contribution is not symmetric under interchanging I and $I+1$, rather one must also change $\mu_I \rightarrow \epsilon_1 + \epsilon_2 - \mu_I$. Taking the NS limit as before gives rise to the following terms in the instanton action $\mathcal{Y}_{I,I+1}(x_j) - \mathcal{Y}_{I,I+1}(x_j^{(0)})$ where

$$\mathcal{Y}_{I,I+1}(x_j) = \sum_{li, nj} \left(f(x_{li}^{(I)} - x_{nj}^{(I+1)} + \mu_I) + f(x_{li}^{(I+1)} - x_{nj}^{(I)} + \epsilon - \mu_I) \right) . \quad (3.2)$$

As previously, the instanton action functional can be written in terms of a set of instanton densities $\rho_I(x)$, $I = 1, \dots, p$, which are constant between the points $[x_{li}^{(I)}, x_{li}^{(0,I)}]$ as

$$\mathcal{H}_{\text{inst}}[\rho_I] = -\frac{1}{2} \int dx dy \rho_I(x) \mathfrak{G}_{IJ}(x-y) \rho_J(y) + \int dx \rho_I(x) \log(q_I \mathfrak{R}_I(x)) , \quad (3.3)$$

where the non-vanishing components of the kernel are

$$\mathfrak{G}_{II}(x) = \frac{d}{dx} \log\left(\frac{x-\epsilon}{x+\epsilon}\right) , \quad \mathfrak{G}_{I,I+1}(x) = \mathfrak{G}_{I+1,I}(-x) = \frac{d}{dx} \log\left(\frac{x+\mu_I}{x-\epsilon+\mu_I}\right) . \quad (3.4)$$

We also define

$$\begin{aligned} \mathfrak{R}_I(x) &= \frac{P_{I-1}(x+\epsilon-\mu_{I-1})P_{I+1}(x+\mu_I)}{P_I(x)P_I(x+\epsilon)} , \quad 1 < I < p , \\ \mathfrak{R}_1(x) &= \frac{A(x)P_2(x+\mu_1)}{P_1(x)P_1(x+\epsilon)} , \quad \mathfrak{R}_p(x) = \frac{P_{p-1}(x+\epsilon-\mu_{p-1})D(x+\epsilon)}{P_p(x)P_p(x+\epsilon)} , \end{aligned} \quad (3.5)$$

where $P_I(x) = \prod_{l=1}^L (x - a_l^{(I)})$. The saddle-point equations are simple to write down. When the quantisation conditions are imposed

$$a_l^{(I)} = m_l - n_l^{(I)}\epsilon - \sum_{J=I}^p \mu_J , \quad \mu_p = 0 , \quad (3.6)$$

one can again show the degeneration of intervals

$$x_{li}^{(I)} = x_{li}^{(0,I)} = a_l^{(I)} + (i-1)\epsilon, \quad i \geq n_l^{(I)}, \quad (3.7)$$

leading to truncation of the saddle-point equations. The root of baryonic Higgs branch in this linear quiver case is now located at

$$a_l^{(I)} = m_l - \epsilon - \sum_{J=I}^p \mu_J, \quad (3.8)$$

where $\mathfrak{R}_I(x)$ in (3.5), or equivalently instanton partition function vanish identically, due to the additional zero modes that pop up the Higgs branch moduli.

Defining again the truncated quantities

$$\hat{\mathfrak{Q}}_I(x) = \prod_{l=1}^L \prod_{i=1}^{n_l^{(I)}-1} (x - x_{li}^{(I)}), \quad (3.9)$$

the saddle point equations become

$$\begin{aligned} -q_1 \frac{\hat{\mathfrak{Q}}_1(x_{li}^{(1)} - \epsilon)}{\hat{\mathfrak{Q}}_1(x_{li}^{(1)} + \epsilon)} \frac{\hat{\mathfrak{Q}}_2(x_{li}^{(1)} + \mu_1)}{\hat{\mathfrak{Q}}_2(x_{li}^{(1)} - \epsilon + \mu_1)} &= \frac{D(x_{li}^{(1)} + \sum_{J=1}^{p-1} \mu_J + 2\epsilon)}{A(x_{li}^{(1)})}, \\ -q_I \frac{\hat{\mathfrak{Q}}_{I-1}(x_{li}^{(I)} + \epsilon - \mu_{I-1})}{\hat{\mathfrak{Q}}_{I-1}(x_{li}^{(I)} - \mu_{I-1})} \frac{\hat{\mathfrak{Q}}_I(x_{li}^{(I)} - \epsilon)}{\hat{\mathfrak{Q}}_I(x_{li}^{(I)} + \epsilon)} \frac{\hat{\mathfrak{Q}}_{I+1}(x_{li}^{(I)} + \mu_1)}{\hat{\mathfrak{Q}}_{I+1}(x_{li}^{(I)} - \epsilon + \mu_I)} &= 1, \quad (1 < I < p) \\ -q_p \frac{\hat{\mathfrak{Q}}_{p-1}(x_{li}^{(p)} + \epsilon - \mu_{p-1})}{\hat{\mathfrak{Q}}_{p-1}(x_{li}^{(p)} - \mu_{p-1})} \frac{\hat{\mathfrak{Q}}_p(x_{li}^{(p)} - \epsilon)}{\hat{\mathfrak{Q}}_p(x_{li}^{(p)} + \epsilon)} &= 1. \end{aligned} \quad (3.10)$$

With the dictionary below

$$\begin{aligned} x^{(I)} &= \lambda^{(I)} - \sum_{J=I}^{p-1} (\mu_J - \frac{1}{2}\epsilon) - \frac{1}{2}\epsilon, \\ M_l &= m_l - \frac{p+2}{2}\epsilon, \quad \tilde{M}_l = \tilde{m}_l + \sum_{J=1}^{p-1} (\mu_J - \frac{1}{2}\epsilon) + \frac{1}{2}\epsilon, \end{aligned} \quad (3.11)$$

the above equations (3.10) are exactly the BAE of an $SL(p+1, \mathbb{R})$ spin chain

$$-q_I \prod_{J=1}^p \frac{Q_J(\lambda_j^{(I)} - \frac{1}{2}\epsilon C_{IJ})}{Q_J(\lambda_j^{(I)} + \frac{1}{2}\epsilon C_{IJ})} = \begin{cases} \frac{d(\lambda_j^{(1)})}{a(\lambda_j^{(1)})} & I = 1 \\ 1 & I > 1, \end{cases} \quad (3.12)$$

where $a(\lambda)$ and $d(\lambda)$ are defined in (2.36) and C_{IJ} is the Cartan matrix of the Lie algebra associated to $SL(p+1)$,

$$C_{IJ} = 2\delta_{IJ} - \delta_{I,J+1} - \delta_{I,J-1} . \quad (3.13)$$

The classical instanton string solutions,

$$x_{li}^{(I,0)} = a_l^{(I)} + (i-1)\epsilon = m_l - n_l^{(I)}\epsilon - \sum_{J=I}^{p-1} \mu_J + (i-1)\epsilon , \quad (3.14)$$

$i = 1, \dots, n_l^{(I)} - 1$, are related to classical Bethe roots

$$\lambda_{(ls)}^{(I,0)} = M_l - (s-1)\epsilon + \frac{I-1}{2}\epsilon , \quad (3.15)$$

by $s = n_l^{(I)} - i$. It is straightforward to show that the instanton action matches the Yang-Yang functional of the spin chain generalizing (2.40) in an obvious way.

Theory II in correspondence is therefore a two dimensional $\mathcal{N} = (2, 2)$ super QCD with quiver gauge group $\prod_{I=1}^p U(N_I)$ with $N_I = \sum_{l=1}^L (n_l^{(I)} - 1)$. Theory II has the matter content of one adjoint hypermultiplet with twisted mass ϵ for each $U(N_I)$, bi-fundamental of twisted mass $\frac{1}{2}\epsilon$ under $U(N_I) \times U(N_{I+1})$, and L fundamental hypermultiplet of masses M_l and anti-fundamental L anti-fundamental of masses \tilde{M}_l under $U(N_1)$. As depicted in Figure (3.2), one can show $n_l^{(I)} - 1 = \sum_{J=I}^p \hat{n}_l^{(J)}$ or equivalently $N_I = \sum_{J=I}^p \sum_{l=1}^L \hat{n}_l^{(J)}$, where $\hat{n}_l^{(J)}$ denotes a number of D2-branes stretched between l^{th} D4-brane and J^{th} NS5-brane. This relation is compatible with an interpretation of the duality in terms of the refined geometric transition proposed in [1].

The twisted superpotential/Yang-Yang functional for Theory II can be quite straightforwardly written down, using the results in [16], and the BAE arising from the F-term on-shell condition is precisely (3.12) above with

$$\hat{q}_I = (-1)^{N_I+1} q_I . \quad (3.16)$$

In order to complete the duality between the two sides, we need to show that the perturbative pieces match; that is,

$$\mathcal{W}_{\text{pert}}^{(\text{I})} \left(m_l - n_l^{(I)}\epsilon - \sum_{J=I}^{p-1} \mu_J \right) - \mathcal{W}_{\text{pert}}^{(\text{I})} \left(m_l - \epsilon - \sum_{J=I}^{p-1} \mu_J \right) = \mathcal{W}^{(\text{II})} (\lambda_j^{(I,0)}) , \quad (3.17)$$

which generalizes (2.46). The matching of the classical contributions is guaranteed by the identity

$$\left(m_l - n_l^{(I)}\epsilon - \sum_{J=I}^{p-1} \mu_J\right)^2 - \left(m_l - \epsilon - \sum_{J=I}^{p-1} \mu_J\right)^2 = -2\epsilon \sum_{s=1}^{n_l^{(I)}-1} \lambda_{(ls)}^{(I,0)} . \quad (3.18)$$

The one-loop contribution is given by

$$\begin{aligned} \mathcal{W}_{1\text{-loop}}^{(\text{I})}(a_l^{(I)}) &= \sum_{ln} [\omega_\epsilon(a_l^{(1)} - \tilde{m}_n) + \omega_\epsilon(a_l^{(p)} - m_n - \epsilon) + \omega_\epsilon(a_l^{(I)} - a_n^{(I)})] \\ &\quad + \sum_{I=1}^{p-1} \sum_{ln} \omega_\epsilon(a_l^{(I)} - a_n^{(I+1)} + \mu_I) \end{aligned} \quad (3.19)$$

and after some tedious work, one can show that

$$\begin{aligned} \mathcal{W}_{1\text{-loop}}^{(\text{I})}\left(m_l - n_l^{(I)}\epsilon - \sum_{J=I}^{p-1} \mu_J\right) - \mathcal{W}_{1\text{-loop}}^{(\text{I})}\left(m_l - \epsilon - \sum_{J=I}^{p-1} \mu_J\right) \\ = \mathcal{W}^{(\text{II})}(\lambda_j^{(I,0)}) - \log q \sum_{Ij} \lambda_j^{(I,0)} , \end{aligned} \quad (3.20)$$

as required, and this completes the proof of the duality for the finite quiver theories.

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